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ANALYTICAL SOLUTION OF THE SATURATED FLOW PROBLEM IN 7-SPOT, 2D GEOMETRIES

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ABSTRACT

The problem of saturated flow within a homogeneous and isotropic pore formation, confined between two horizontal impermeable planes, under 7-spot injection-extraction well pattern is considered. Such well patterns are typically implemented in soil remediation or enhanced oil recovery processes. Extraction wells (rectilinear sinks) are uniformly distributed over the reservoir domain, creating a honeycomb pattern of identical hexagons. An injection well (rectilinear source) is located at the centre of each hexagon. Uniform strength is considered for all sources. In that context, the flow within every hexagon can be partitioned into identical flows in each of the six equilateral triangles. To furnish the analytical expressions for the pressure and velocity fields, we have to solve an interior Neumann problem for the Laplace equation, considering that the normal derivative of the pressure is known on the boundary of the equilateral triangle. To deal with this unconventional geometry (the method of separation of variables is not applicable) we implement the new method provided by Dassios and Fokas in [1], whereby the authors study boundary value problems for the Laplace, the Helmholtz and the modified Helmholtz equations in the interior of an equilateral triangle.

KEYWORDS:
saturated flow, porous media, 7-spot

INTRODUCTION

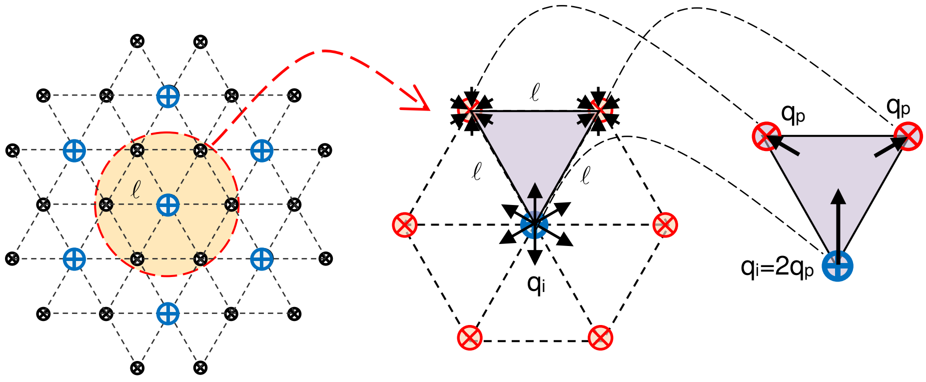
Saturated flow in porous media is a core physical process with many engineering applications, mainly related to groundwater management (capture zones in pumping wells, salt-water intrusion in coastal aquifers, replenishment), seepage analysis of dams, discharge of wells near rivers, etc. The basic phenomenology of saturated flow is described by Darcy's law, explicitly providing a linear relation between the superficial velocity (volumetric flowrate intensity) and the field pressure gradient, whereby the linearity constant is the ratio of the porous medium absolute permeability to fluid (water) viscosity. A variety of problems can be treated, either analytically or semi-analytically as long as these can be described in typical geometries. The solution of problems in more sophisticated or atypical geometries requires the implementation of appropriate numerical schemes. A collection of problems treated with various methodological approaches can be found in [2, 3].

An equally important physical process, in terms of engineering applications, is the unsaturated flow, or immiscible two-phase flow, in porous media. Indicative applications can be found in irrigation and drainage, i.e. the simultaneous flow of water and air, and in pollution of aquifers and the associated remediation interventions, water flooding of oil reservoirs, enhanced oil recovery, CO_2 sequestration etc. i.e. the simultaneous flow of a wetting and a non-wetting phase. The governing equation for unsaturated (or two-phase) flow in porous media is produced by extending the conventional Darcy law to the fractional Darcy law, by introducing the concept of the effective permeability of each phase to account for the hydrodynamic and capillary coupling observed during the simultaneous flow of both phases. Conventionally, effective permeabilities are treated as functions of the saturation. Nevertheless, there are many inadequacies in correctly describing the flow with this approach. The recently developed mechanistic model *DeProF* and corresponding theory for steady-state two-phase flow in porous media [4] has derived a universal scaling law for the reduced pressure gradient in terms of the actual independent variables of the process, i.e. the local values of the superficial velocity of oil and water [5, 6].

Now, with the availability of an explicit scaling law describing the process, the associated two-phase (unsaturated) flow problem can be transformed into an equivalent one-phase (saturated) flow problem.

To this end any development of analytical solutions for the saturated flow problem in difficult, non-trivial or atypical geometries, would enhance our capability of understanding the flow response to various configurations and to design more efficient processes. One such atypical geometrical configuration is the 7-spot injection-extraction pattern [2, 7].

The 7-spot pattern flow equivalence analysis. The case of the 7-spot pattern flow arrangement



(a) (b) (c)

FIGURE 1

Modular break-down of a 7-spot pattern well pattern formation (a) **Layout of the 7-spot formation of injection wells ("sources", ⊕) and water production wells ("sinks",⊗). The honeycomb pattern extends infinitely in both directions.** (b) **The 7-spot pattern hexagonal building block** (c) **the modular unit cell (equilateral triangle).**

within a homogeneous and isotropic porous medium will be considered. A schematic representation of the 7-spot pattern is provided in Figure 1, whereby production wells (rectilinear sinks) are uniformly distributed over a reservoir field and create a honeycomb pattern of identical right hexagonal prisms, with lattice constant, ℓ. An injection well (rectilinear source) is located at the axis of each hexagonal prism. If the whole space is filled with these identical prisms (7-spot modules), then it can be shown (by inductive reasoning) that in the asymptotic limit, the ratio of sources to sinks is ½, i.e. the number of sinks is twice the number of sources. In the present work, ⊕ indicates a point source and ⊗ indicates a point sink. Because of the porous medium homogeneity and isotropy and the geometrical symmetry, we can postulate that solving the problem in the infinitely extending 7-spot layout (depicted in Figure 1) is equivalent to solving the problem within an equilateral triangle with impervious sides. Suppose the strength of any source (specific flowrate or volume flowrate per unit source legth) is q_i and, similarly, the strength of any sink is q_p. The dimensions of both strengths are L^2T^{-1}. Because of the 7-spot geometrical symmetry, to find the flowrate per source (or sink) unit length *within* the isolated equilateral triangle (with *impervious sides*), volume balance suggests that $q_i/6 = 2q_p/6$, therefore $2q_p = q_i$.

Analytical/mathematical implications in triangular geometries. It is well understood that solving analytically the Laplace equation inside a triangle (given any type of Dirichlet, Neumann, Robin or mixed type boundary conditions) is not possible because there is no coordinate system matching the triangular geometry configuration upon which, when expressed, the Laplace partial differential equation can be separated into two (for 2D problems) independent ordinary differential equations.

Nevertheless, the particular case of the Dirichlet (or Neumann) problem for the solution of the Laplace equation within an equilateral triangle can be handled in an analytical manner, using the novel method introduced by Dassios and Fokas [1]. In their paper Dassios and Fokas manage the so called, "global relation" and present a procedure for the solution of boundary value problems for the Laplace, the Helmholtz and the modified Helmholtz equations in the interior of an equilateral triangle. So far there have been two applications of the Dassios & Fokas method. In [8], the Laplace equation was solved in an exterior non-convex domain which is the Kelvin image of an equilateral triangle- subject to Neumann boundary condition, while in [9] Baganis and Hadjinicolaou derive an analytic solution of the corresponding Dirichlet problem.

Here we present a methodological roadmap one has to follow in order to derive an explicit analytical solution for the sought problem, i.e. interior Neumann problem for the Laplace equation.

STATEMENT OF THE PROBLEM

Consider a point source with strength Q at $\boldsymbol{r}_3 = \left(-\ell/\sqrt{3}, 0\right)$ and a pair of sinks with equal strength $-Q/2$, at positions $\boldsymbol{r}_1 = \left(\ell\sqrt{3}/6, \ell/2\right)$ and $\boldsymbol{r}_2 = \left(\ell\sqrt{3}/6, -\ell/2\right)$. The vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ correspond to the complex numbers z_1, z_2, z_3, as introduced in [1] and define the vertices of the equilateral triangle (the length of each side is ℓ, see Figure 2. Following [1], we denote, also, the sides $(z_2, z_1), (z_3, z_2)$

and (z_1, z_3), as sides (1), (2) and (3) respectively.

In the interior of the triangle D, the pressure field assumes the following form [In fact, the pressure field incorporates the constant factor, $(-k/\mu)$, with k the absolute permeability of the porous medium and μ the dynamic viscosity of the fluid]

$$p(\boldsymbol{r}) = \frac{Q}{2\pi}\ln|\boldsymbol{r}-\boldsymbol{r}_3| - \frac{Q}{4\pi}\ln|\boldsymbol{r}-\boldsymbol{r}_1| - \frac{Q}{4\pi}\ln|\boldsymbol{r}-\boldsymbol{r}_2| + q(\boldsymbol{r}) \quad , \quad \boldsymbol{r}\in D \tag{1}$$

where $q(\boldsymbol{r})$ is a harmonic function in D, while on the boundary ∂D, as the sides of the triangle are impervious, the normal component of the velocity field must be zero and the following Neumann condition must be imposed

$$\frac{\partial p(\boldsymbol{r})}{\partial n} = 0 \quad , \quad \boldsymbol{r}\in\partial D \tag{2}$$

Eqn (1) express the contribution of the fundamental solution of two-dimensional Laplace equation while eqn (2) yields the Neumann data $q_N^{(j)}(s)$, $j=1,2,3$ on each side of the triangle.

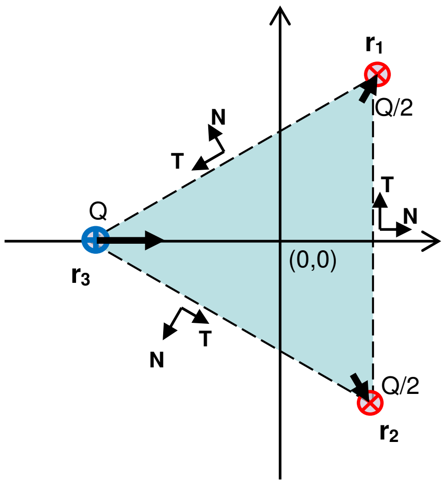


FIGURE 2
The fundamental domain *D*.

Therefore, we have to solve an interior Neumann problem for the Laplace equation and the well-known compatibility condition (representing the mass balance within D)

$$\oint_{\partial D} \frac{\partial q(\boldsymbol{r})}{\partial n} dl(\boldsymbol{r}) = 0 \tag{3}$$

must be valid for a solution to exist.

With respect to the same parameter $s\in[-\ell/2, \ell/2]$, the following set of vector parametric representations, $\boldsymbol{r}(s)$, and unit normals, $\hat{\boldsymbol{N}}$, are considered on the 3 sides of the equilateral triangle, see Figure 2:

Side (1)

$$\boldsymbol{r}(s) = \left(\frac{\ell}{2\sqrt{3}}, s\right), \ \hat{\boldsymbol{N}} = (1,0)$$

Side (2)

$$\boldsymbol{r}(s) = \left(-\frac{\ell}{4\sqrt{3}} + \frac{s\sqrt{3}}{2}, -\frac{\ell}{4} - \frac{s}{2}\right), \ \hat{\boldsymbol{N}} = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \tag{4}$$

Side (3)

$$\boldsymbol{r}(s) = \left(-\frac{\ell}{4\sqrt{3}} - \frac{s\sqrt{3}}{2}, \frac{\ell}{4} - \frac{s}{2}\right), \ \hat{\boldsymbol{N}} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

The first step in implementing the method is the evaluation of the Neumann data $q_N^{(j)}(s)$, $j=1,2,3$, with respect to s, that define the new functions

$$F_j(k) = \frac{1}{2}\int_{-\ell/2}^{\ell/2} e^{ks} q_N^{(j)}(s)ds \quad , \quad j=1,2,3 \tag{5}$$

Next, one has to recover the unknown Dirichlet data $q^{(j)}(s)$, $j=1,2,3$, on the boundary ∂D. In [1, Proposition 3.3] the authors establish the so-called *Neumann-to-Dirichlet map* for the generalized Helmholtz equation. In the specific case of Laplace equation we can simplify the result by straightforward calculations as follows.

Proposition 2.1. Let the real valued function $q(x,y)$ satisfy the Laplace equation in D, with Neumann boundary conditions

$$q_N^{(j)}(s) = f_j(s) \quad , \quad j=1,2,3 \tag{6}$$

whereby the known functions f_j satisfy the compatibility condition

$$\int_{-\ell/2}^{\ell/2} [f_1(s) + f_2(s) + f_3(s)]ds = 0 \tag{7}$$

Then, the Dirichlet data $q^{(j)}(s)$, $j=1,2,3$ can be expressed in terms of the known Neumann data by the Fourier series

$$q^{(j)}(s) = \sum_{n=-\infty}^{n=+\infty} \begin{bmatrix} N(k_{3n}) \\ +c_1^{(j)} e^{2i\pi s/3\ell} N(k_{3n-1}) \\ +c_2^{(j)} e^{4i\pi s/3\ell} N(k_{3n-2}) \end{bmatrix} e^{-2i\pi ns/\ell} \tag{8}$$

where

$$c_1^{(1)} = c_2^{(1)} = 1, \quad c_1^{(2)} = c_2^{(3)} = \bar{a}, \quad c_1^{(3)} = c_2^{(2)} = a,$$
$$a = e^{i2\pi/3}, \quad \bar{a} = e^{-i2\pi/3} \tag{9}$$

and

$$N(k_m)=\frac{2}{m\pi\left[e^{m\pi/\sqrt{3}}+(-1)^{m+1}\right]}A \qquad (10)$$

where

$$A=\begin{Bmatrix} e^{(\sqrt{3}+3i)\frac{m\pi}{6}}\cos\left(\frac{3+i\sqrt{3}}{6}m\pi\right)F_1(k_m) \\ +e^{(\sqrt{3}+5i)\frac{m\pi}{6}}\cos\left(\frac{3-i\sqrt{3}}{6}m\pi\right)F_2(k_m) \\ +e^{(\sqrt{3}+7i)\frac{m\pi}{6}}\cos\left(\frac{3+i\sqrt{3}}{6}m\pi\right)F_3(k_m) \\ +e^{(\sqrt{3}+9i)\frac{m\pi}{6}}F_1(ak_m)+e^{(\sqrt{3}+5i)\frac{m\pi}{6}}F_2(ak_m) \\ +e^{(\sqrt{3}+i)\frac{m\pi}{6}}F_3(ak_m)+e^{(\sqrt{3}+3i)\frac{m\pi}{6}}F_1(\bar{a}k_m) \\ +(-1)^m e^{(\sqrt{3}+5i)\frac{m\pi}{6}}F_2(\bar{a}k_m)+e^{(\sqrt{3}+7i)\frac{m\pi}{6}}F_3(\bar{a}k_m) \end{Bmatrix} \qquad (11)$$

$$k_m=\frac{2im\pi}{3\ell} \quad , \quad m\in\mathbf{Z} \qquad (12)$$

and

$$F_j(k) \quad , \qquad j=1,2,3 \qquad (13)$$

are given in [5].

We are ready now to apply the above proposition to the problem at hand.

DETERMINATION OF THE UNKNOWN DIRICHLET DATA

In view of (2), straightforward differentiation of (1) yields the Neumann data

$$q_N^{(1)}(s)=\frac{-\sqrt{3}\ell Q}{4\pi}\frac{1}{s^2+3\ell^2/4}$$
$$q_N^{(2)}(s)=q_N^{(3)}(s)=\frac{\sqrt{3}\ell Q}{8\pi}\frac{1}{s^2+3\ell^2/4} \qquad (14)$$

and the compatibility condition (7) is satisfied. We define the functions

$$F_1(k)=\frac{-\sqrt{3}\ell Q}{8\pi}G(k)$$
$$F_2(k)=F_3(k)=\frac{\sqrt{3}\ell Q}{16\pi}G(k) \qquad (15)$$

where

$$G(k)=\int_{-\ell/2}^{\ell/2}\frac{e^{ks}}{s^2+\frac{3\ell^2}{4}}ds \qquad (16)$$

Then, the Dirichlet data on each side of the triangle can be expressed by the series

$$q^{(j)}(s)=\sum_{n=-\infty}^{n=+\infty}\begin{bmatrix} c_1^{(j)}N(k_{3n-1})e^{-2i\pi s\frac{3n-1}{3\ell}} \\ +c_2^{(j)}N(k_{3n-2})e^{-2i\pi s\frac{3n-2}{3\ell}} \end{bmatrix} \qquad (17)$$

where

$$c_1^{(1)}=c_2^{(1)}=1,\ c_1^{(2)}=c_2^{(3)}=\bar{a},\ c_1^{(3)}=c_2^{(2)}=a, \quad a=e^{i2\pi/3},\ \bar{a}=e^{-i2\pi/3} \qquad (18)$$

and

$$N(k_m)=\frac{\ell Q\sqrt{3}}{8m\pi^2\left[e^{\frac{m\pi}{\sqrt{3}}}+(-1)^{m+1}\right]}B \qquad (19)$$

where

$$B=\begin{bmatrix} -2e^{(\sqrt{3}+3i)\frac{m\pi}{6}}\cos\left(\frac{3+i\sqrt{3}}{6}m\pi\right) \\ +e^{(\sqrt{3}+5i)\frac{m\pi}{6}}\cos\left(\frac{3-i\sqrt{3}}{6}m\pi\right) \\ +e^{(\sqrt{3}+7i)\frac{m\pi}{6}}\cos\left(\frac{3+i\sqrt{3}}{6}m\pi\right) \end{bmatrix}G(k_m)$$
$$+\left[-2e^{(\sqrt{3}+9i)\frac{m\pi}{6}}+e^{(\sqrt{3}+5i)\frac{m\pi}{6}}+e^{(\sqrt{3}+i)\frac{m\pi}{6}}\right]G(ak_m) \qquad (20)$$
$$+\left[-2e^{(\sqrt{3}+3i)\frac{m\pi}{6}}+(-1)^m e^{(\sqrt{3}+5i)\frac{m\pi}{6}}+e^{(\sqrt{3}+7i)\frac{m\pi}{6}}\right]G(\bar{a}k_m)$$

The harmonic function $q(\boldsymbol{r})$ enjoys the classical integral representation in D

$$q(\boldsymbol{r})=\oint_{\partial D}\begin{bmatrix} -\frac{1}{2\pi}\ln|\boldsymbol{r}-\boldsymbol{r}'|\frac{\partial q(\boldsymbol{r}')}{\partial n'} \\ +q(\boldsymbol{r}')\frac{\partial}{\partial n'}\left(\frac{1}{2\pi}\ln|\boldsymbol{r}-\boldsymbol{r}'|\right) \end{bmatrix}d\ell(\boldsymbol{r}') \ , \quad \boldsymbol{r}\in D \qquad (21)$$

where the integration is taken in the positive direction, $\partial D/\partial n'$ denotes the outward normal derivative on ∂D and $dl(\boldsymbol{r}')$ is the line element along each side.

In view of (14) and (17), the above integral representation provides the analytical expression for the harmonic function $q(\boldsymbol{r})$ or $q(x,y)$

$$q(x,y)=\frac{\ell\sqrt{3}}{8\pi}Q\times$$
$$\times\int_{-\ell/2}^{\ell/2}\frac{2K_1(x,y,s)-K_2(x,y,s)-K_3(x,y,s)}{s^2+3l^2/4}ds$$
$$+\sum_{n=-\infty}^{n=+\infty}\left[N(k_{3n-1})\int_{-\ell/2}^{\ell/2}\begin{bmatrix} E_1(x,y,s) \\ +e^{-\frac{2\pi i}{3}}E_2(x,y,s) \\ +e^{\frac{2\pi i}{3}}E_3(x,y,s) \end{bmatrix}e^{-2i\pi s\frac{3n-1}{3\ell}}ds\right] \qquad (22)$$
$$+\sum_{n=-\infty}^{n=+\infty}\left[N(k_{3n-2})\int_{-\ell/2}^{\ell/2}\begin{bmatrix} E_1(x,y,s) \\ +e^{\frac{2\pi i}{3}}E_2(x,y,s) \\ +e^{-\frac{2\pi i}{3}}E_3(x,y,s) \end{bmatrix}e^{-2i\pi s\frac{3n-2}{3\ell}}ds\right]$$

where the functions,

$$K_1(x,y,s)=\frac{1}{4\pi}\ln\left[\left(x-\frac{\ell}{2\sqrt{3}}\right)^2+(y-s)^2\right]$$
$$K_2(x,y,s)=\frac{1}{4\pi}\ln\left[\left(x+\frac{\ell}{4\sqrt{3}}-s\frac{\sqrt{3}}{2}\right)^2+\left(y+\frac{\ell}{4}+\frac{1}{2}s\right)^2\right] \tag{23}$$
$$K_3(x,y,s)=\frac{1}{4\pi}\ln\left[\left(x+\frac{\ell}{4\sqrt{3}}+s\frac{\sqrt{3}}{2}\right)^2+\left(y-\frac{\ell}{4}+\frac{1}{2}s\right)^2\right]$$

correspond to the value of the fundamental solution of the Laplace equation, while the functions

$$E_1(x,y,s)=\frac{\frac{1}{2\pi}\left(-x+\frac{\ell}{2\sqrt{3}}\right)}{\left(x-\frac{\ell}{2\sqrt{3}}\right)^2+(y-s)^2}$$
$$E_2(x,y,s)=\frac{\frac{1}{2\pi}\left(x+y\sqrt{3}+\frac{\ell}{\sqrt{3}}\right)}{\left(x+\frac{\ell}{4\sqrt{3}}-s\frac{\sqrt{3}}{2}\right)^2+\left(y+\frac{\ell}{4}+\frac{1}{2}s\right)^2} \tag{24}$$
$$E_3(x,y,s)=\frac{\frac{1}{2\pi}\left(x-y\sqrt{3}+\frac{\ell}{\sqrt{3}}\right)}{\left(x+\frac{\ell}{4\sqrt{3}}+s\frac{\sqrt{3}}{2}\right)^2+\left(y-\frac{\ell}{4}+\frac{1}{2}s\right)^2}$$

correspond to the value of its normal derivative on sides (1), (2) and (3) respectively.

CONCLUSIONS

In order to determine the pressure field for the case of saturated, 7-spot pattern flow configuration within a porous medium, we presented a methodology for solving this unconventional flow problem in an analytical fashion.

We first took into account the hexagonal symmetry and the porous medium isotropy and homogeneity to study the equivalent flow problem within any isolated building block, i.e. an equilateral triangle.

Then, we formulated an interior Neumann problem for the unknown harmonic function q, which incorporates the contribution of all sources and sinks placed outside the isolated triangle. To this end, we have first established the Neumann-to-Dirichlet map, since the value of the solution on the boundary of the fundamental domain is involved into its integral representation. Then, the corresponding integral representation provides an analytical expression of the solution in terms of Neumann and Dirichlet data.

In a future work, we need to evaluate the integrals in (22), using complex analysis and simplify this expression as much as possible.

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